دانشگاه آزاد اسلامی واحد تبریز

نام درس: طراحی الگوریتم ها

بخش:

Binary Search Trees and Red-Black Trees

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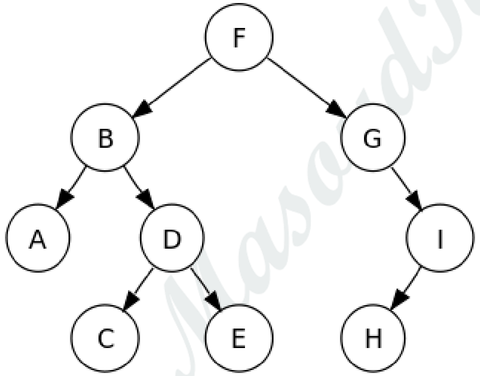
Today: binary search trees

- Brief foray into data structures!
  - See CS 166 for more!
- What are binary search trees?
  - You may remember these from CS 106B
  - Why are they good?
  - Why are they bad?



this will lead us to…

- Self-Balancing Binary Search Trees
  - **Red-Black** trees.

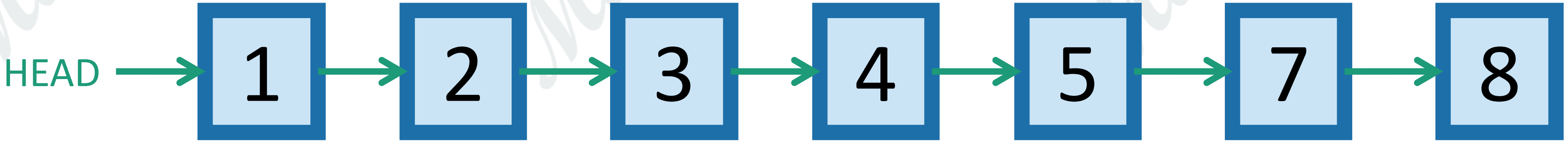
Why are we studying self-balancing BSTs?

1. The punchline is **important**:
   - A data structure with O(log(n)) INSERT/DELETE/SEARCH

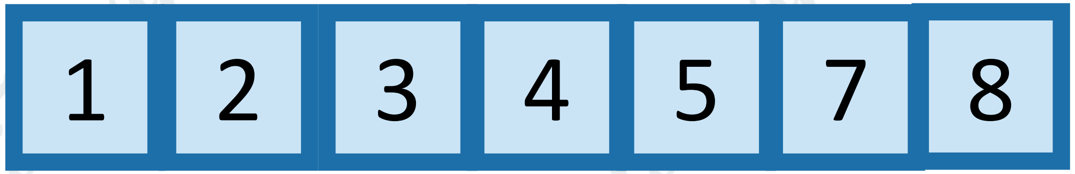
2. The idea behind **Red-Black Trees** is clever
   - It's good to be exposed to clever ideas.
   - Also it's just aesthetically pleasing.

Motivation for binary search trees

- We've been assuming that we have access to some basic data structures:
  - (Sorted) linked lists

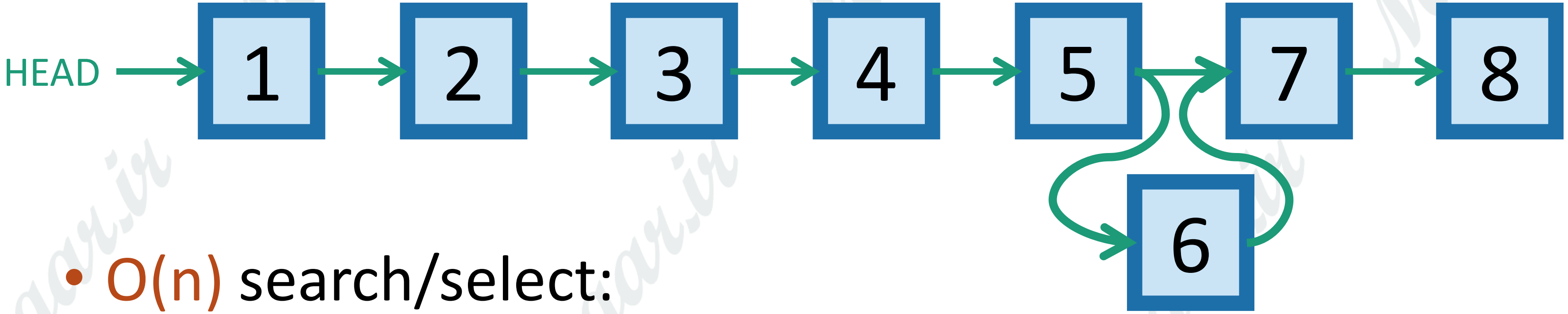


  - (Sorted) arrays

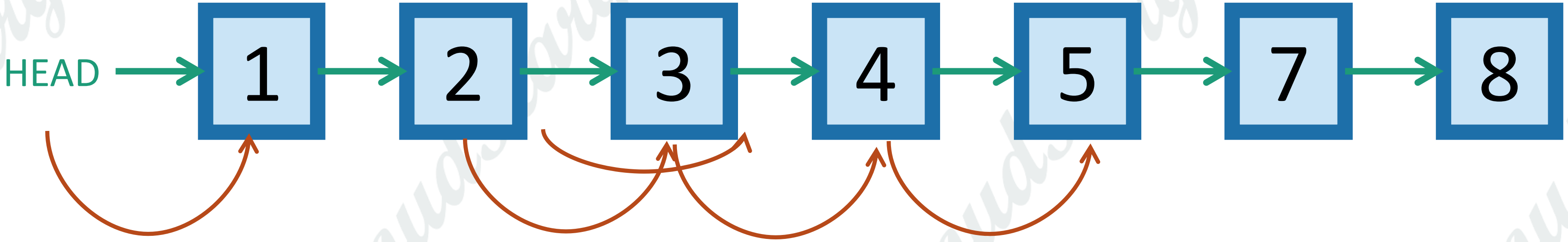


Sorted linked lists

- O(1) insert/delete (assuming we have a pointer to the location of the insert/delete):

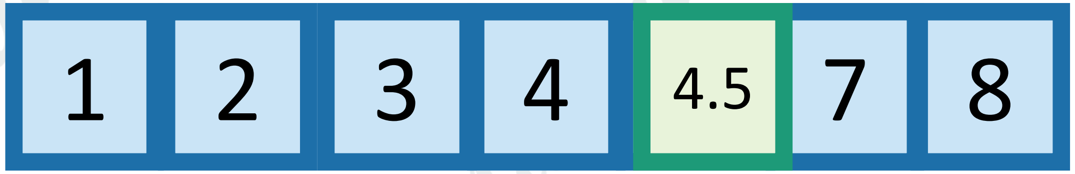


- O(n) search/select:

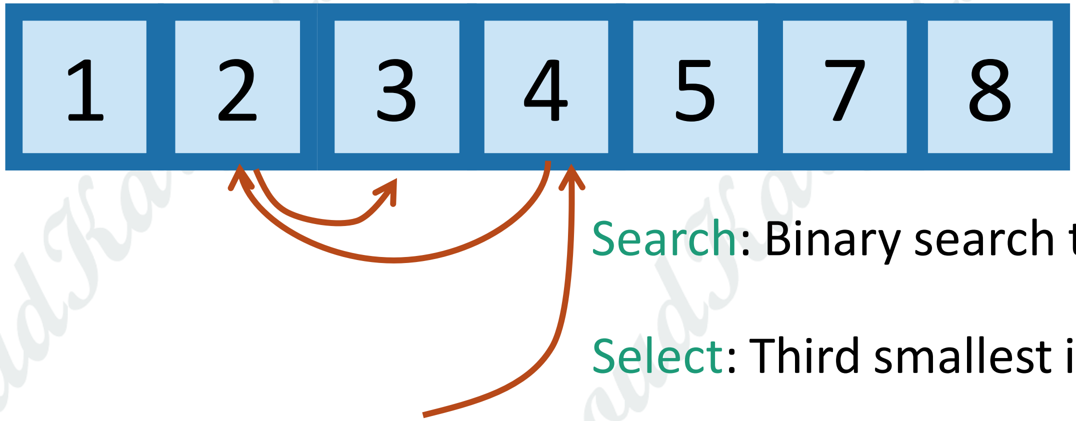


Sorted Arrays

- O(n) insert/delete:



- O(log(n)) search, O(1) select:



Search: Binary search to see if 3 is in A.

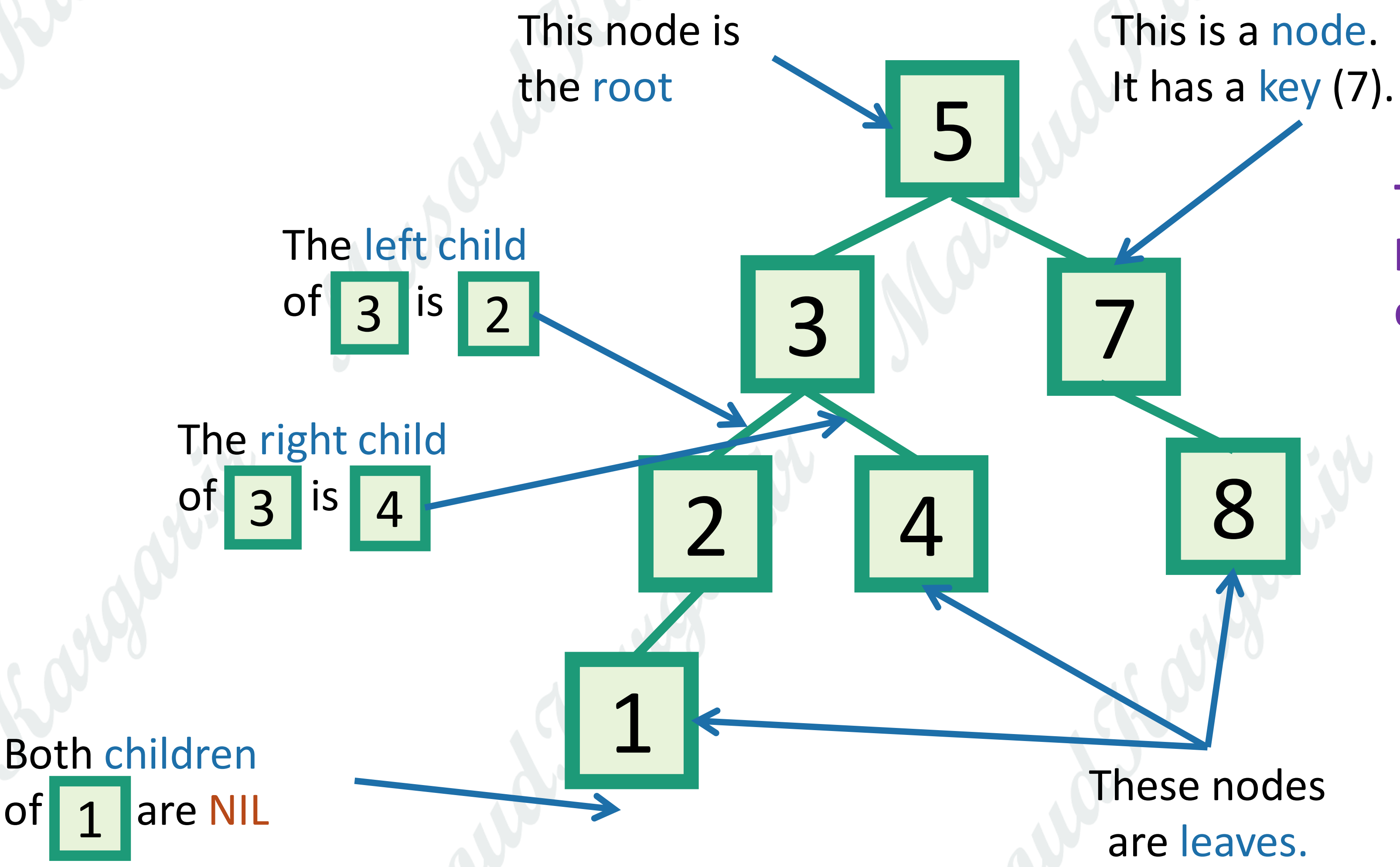
Select: Third smallest is A[3].

The best of both worlds

	Sorted Arrays	Linked Lists	Binary Search Trees*
Search	O(log(n)) 😃	O(n) ☹	O(log(n)) 😃
Insert/Delete	O(n) ☹	O(1) 😃	O(log(n)) 😃

TODAY!

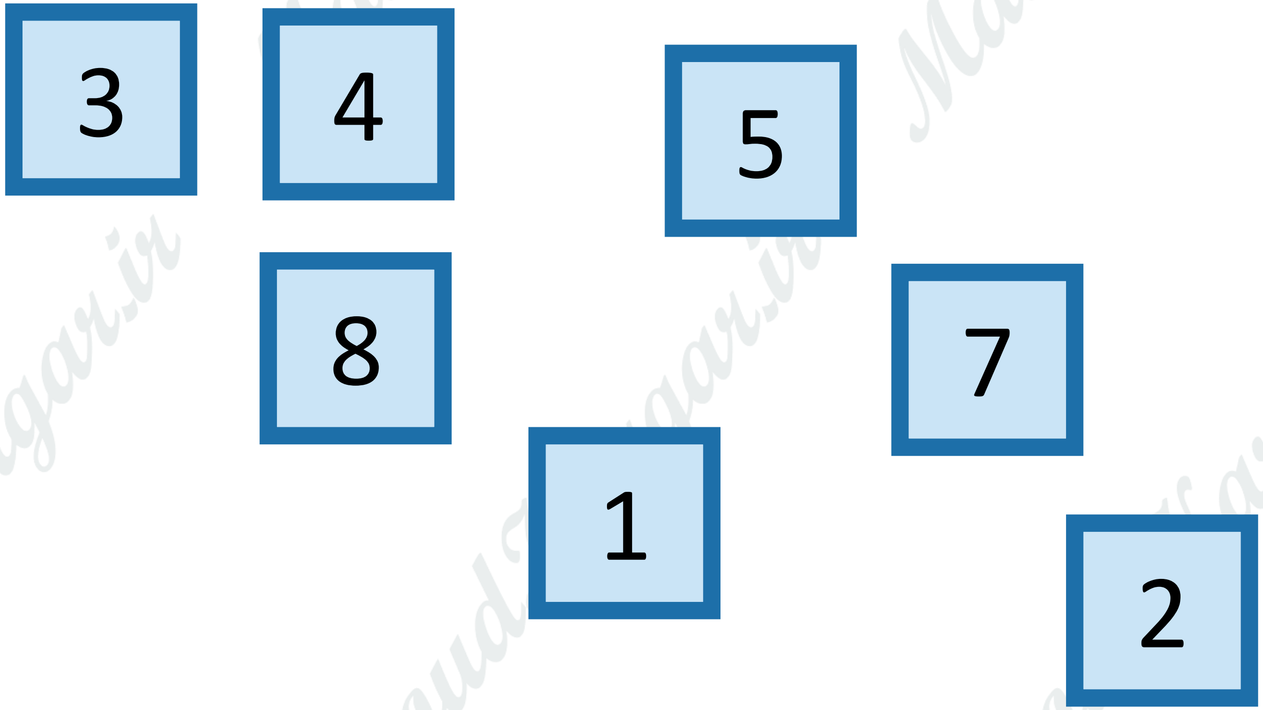
Tree terminology



Today all keys are distinct.

Binary Search Trees

- It’s a binary tree so that:
  - Every LEFT descendent of a node has key less than that node.
  - Every RIGHT descendent of a node has key larger than that node.
- Example of building a binary search tree:



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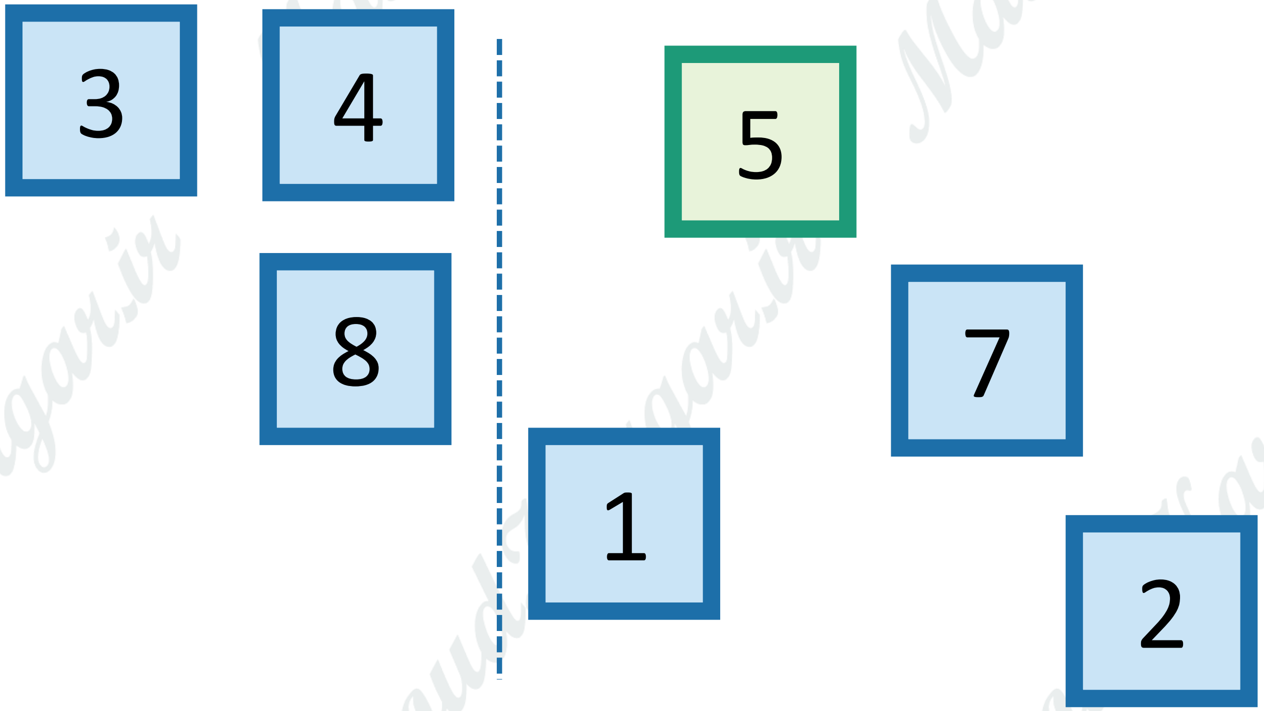

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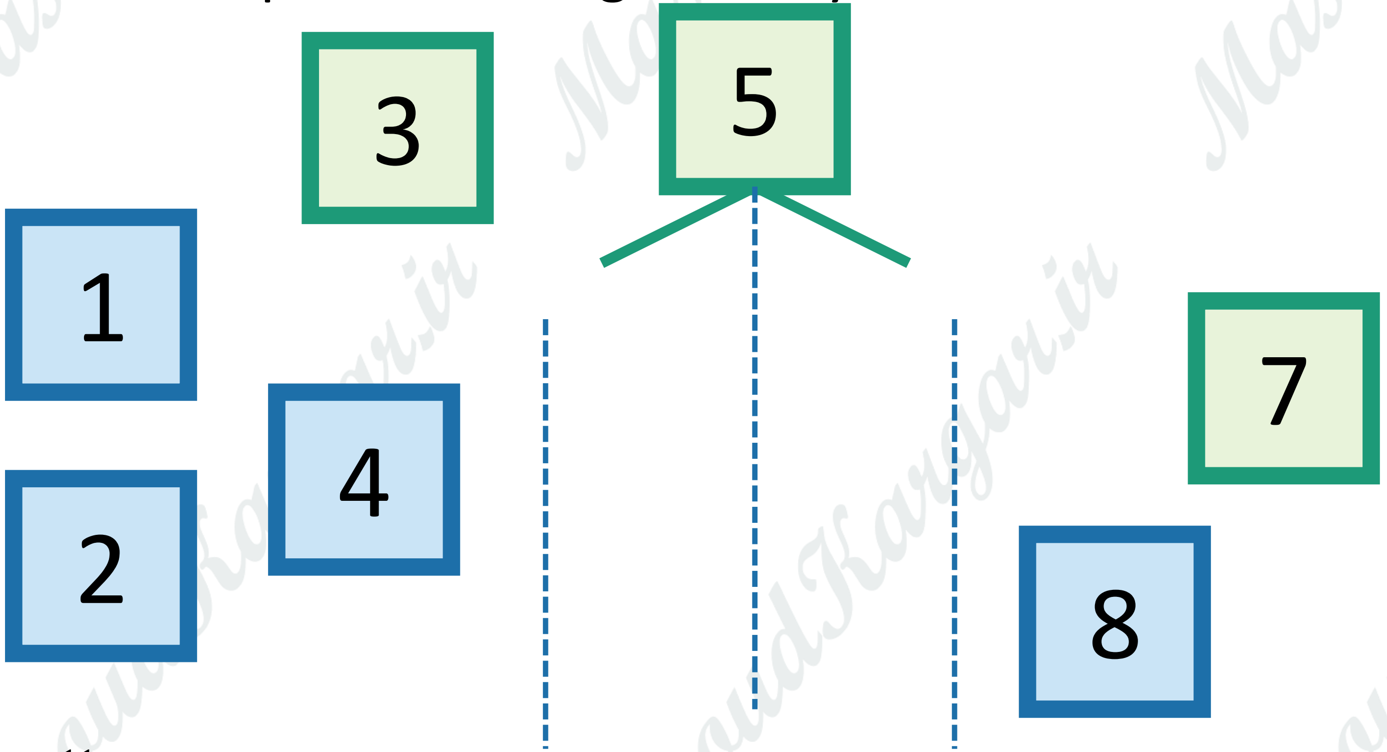

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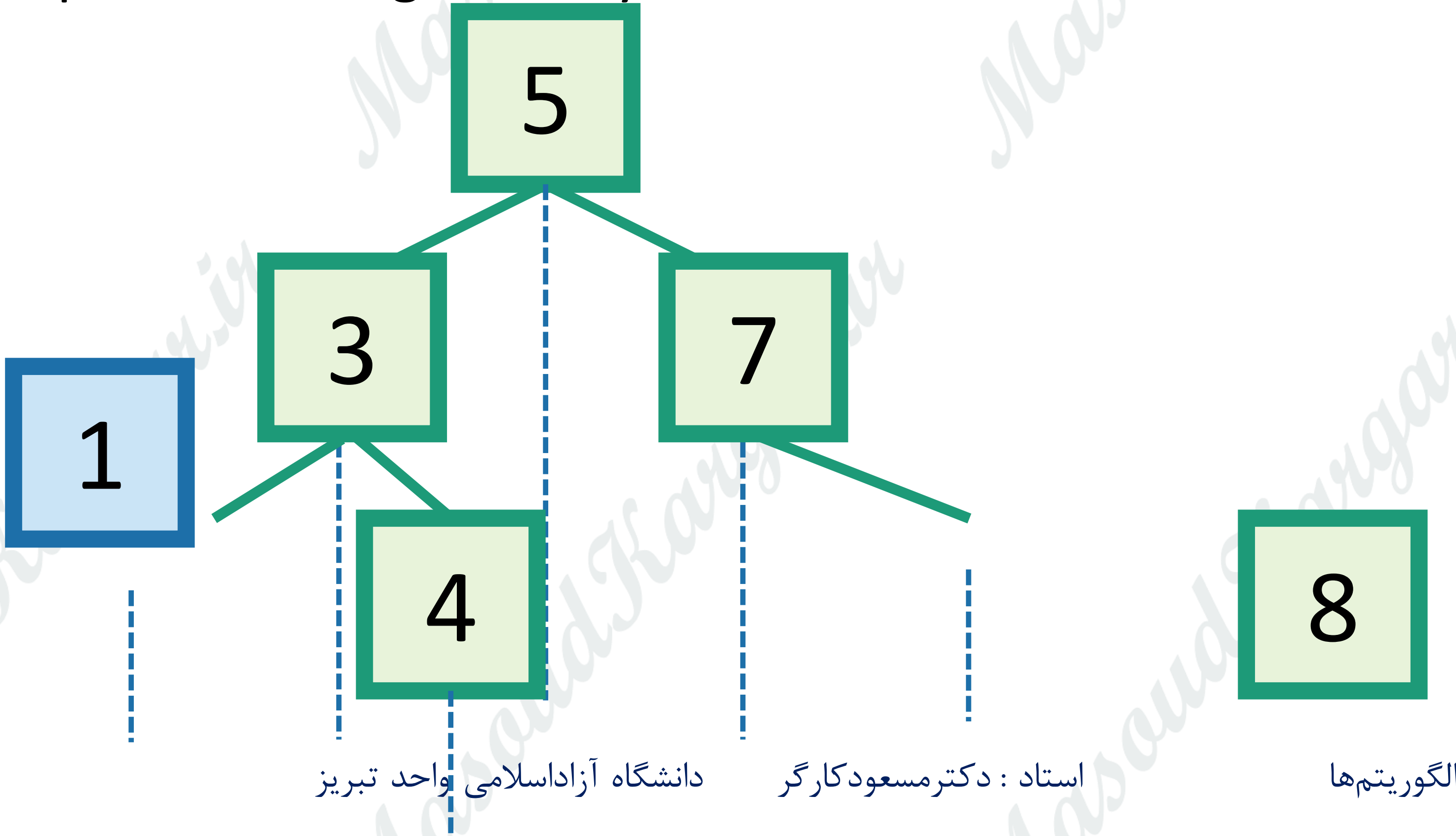

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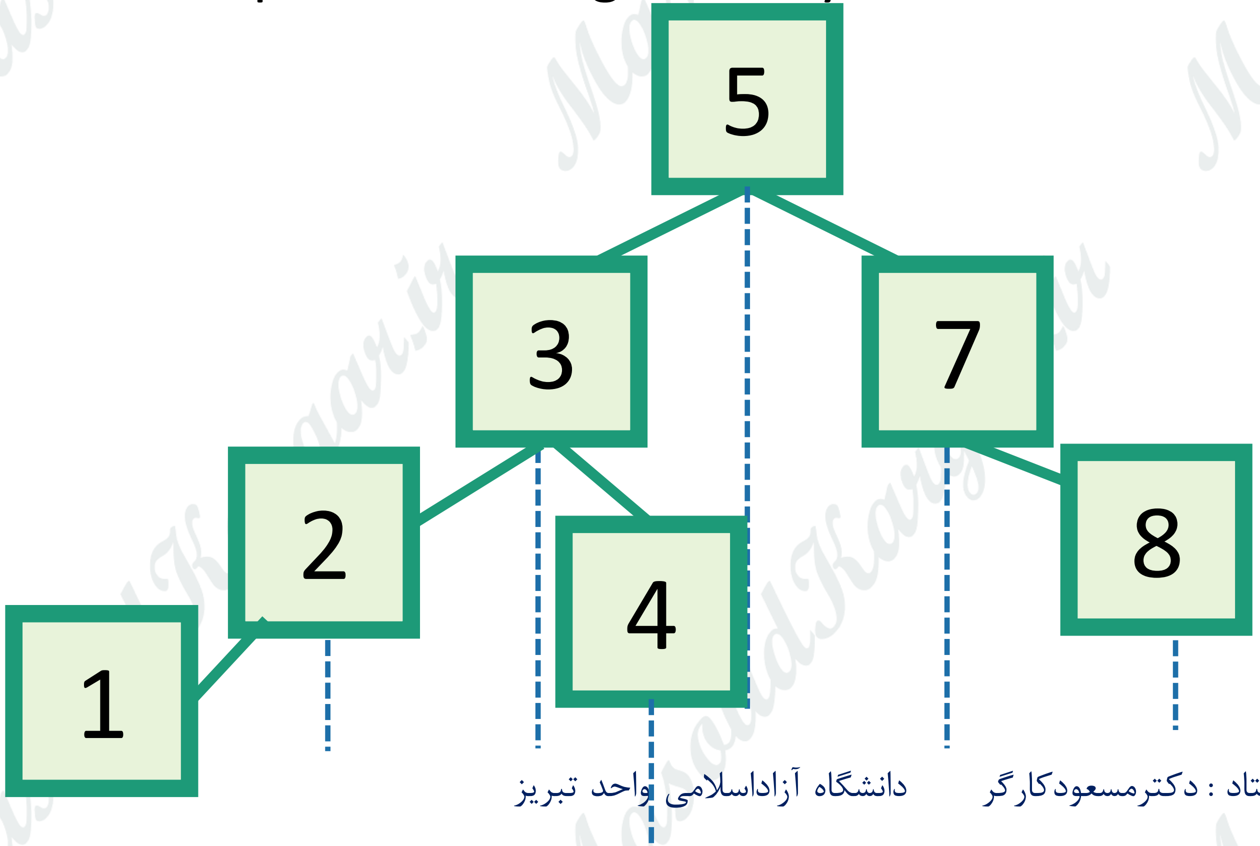
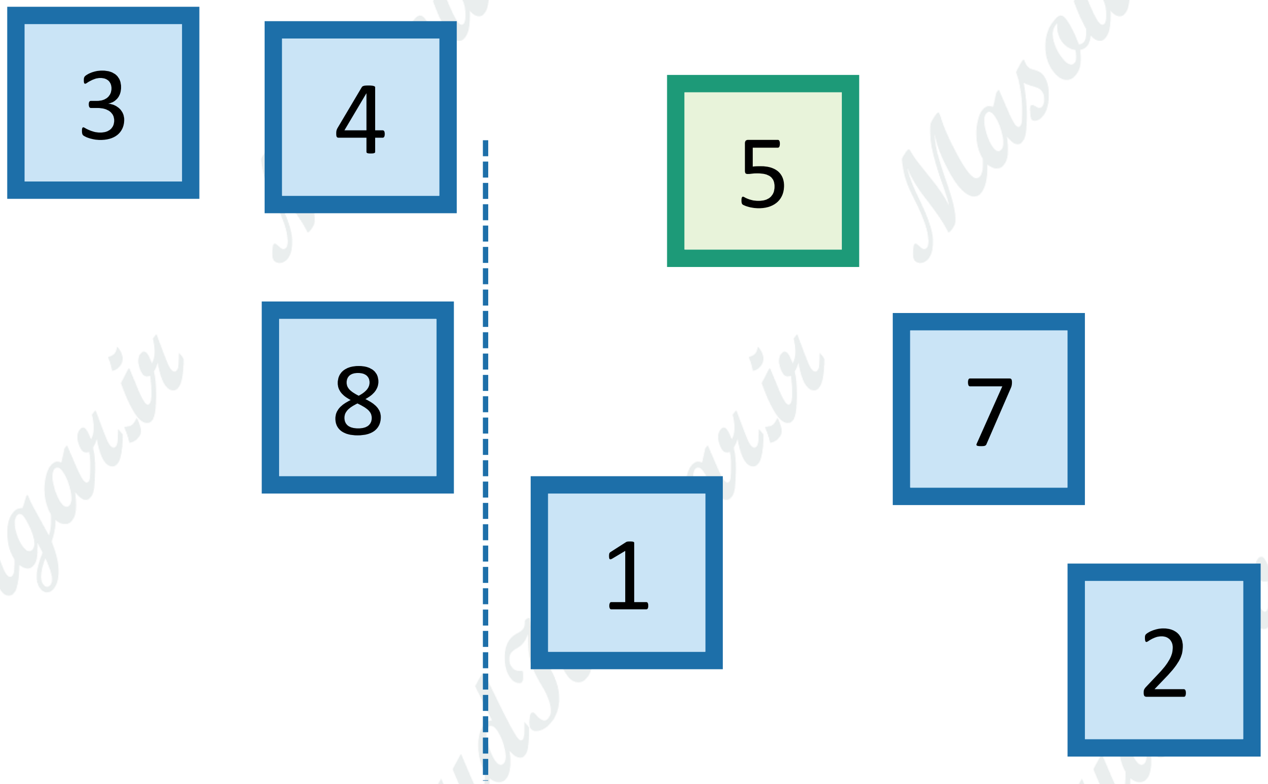


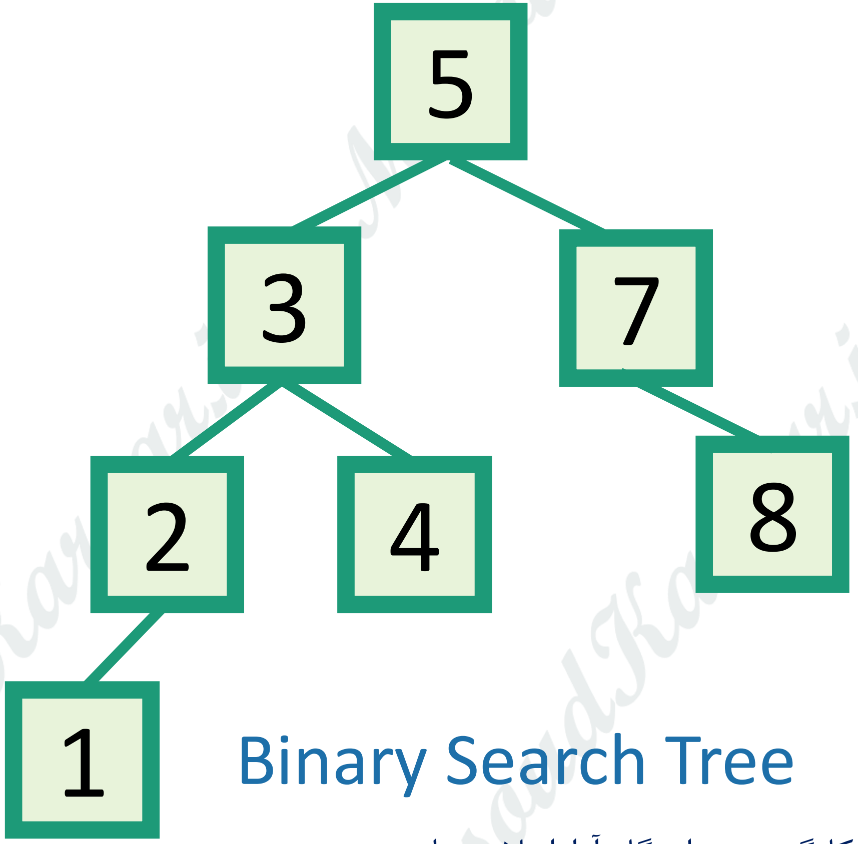
Aside: this should look familiar

kinda like QuickSort



Binary Search Trees

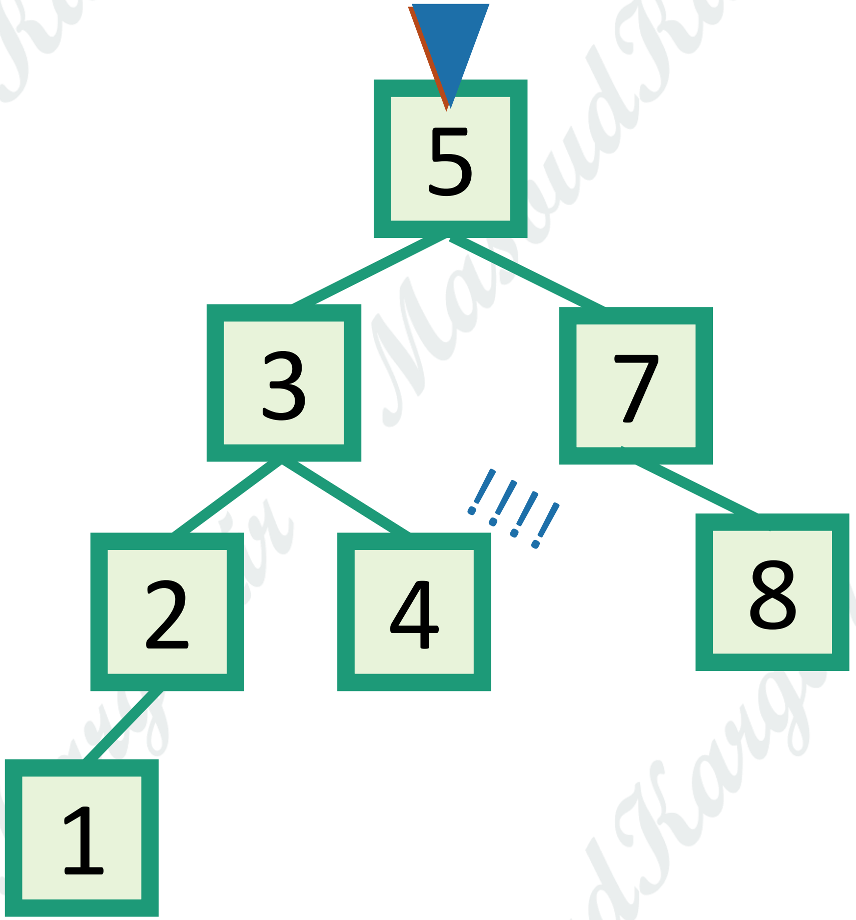
- It's a binary tree so that:
  - **Every LEFT descendent** of a node has key less than that node.
  - **Every RIGHT descendent** of a node has key larger than that node.



Binary Search Tree

NOT a Binary Search Tree

SEARCH in a Binary Search Tree
definition by example



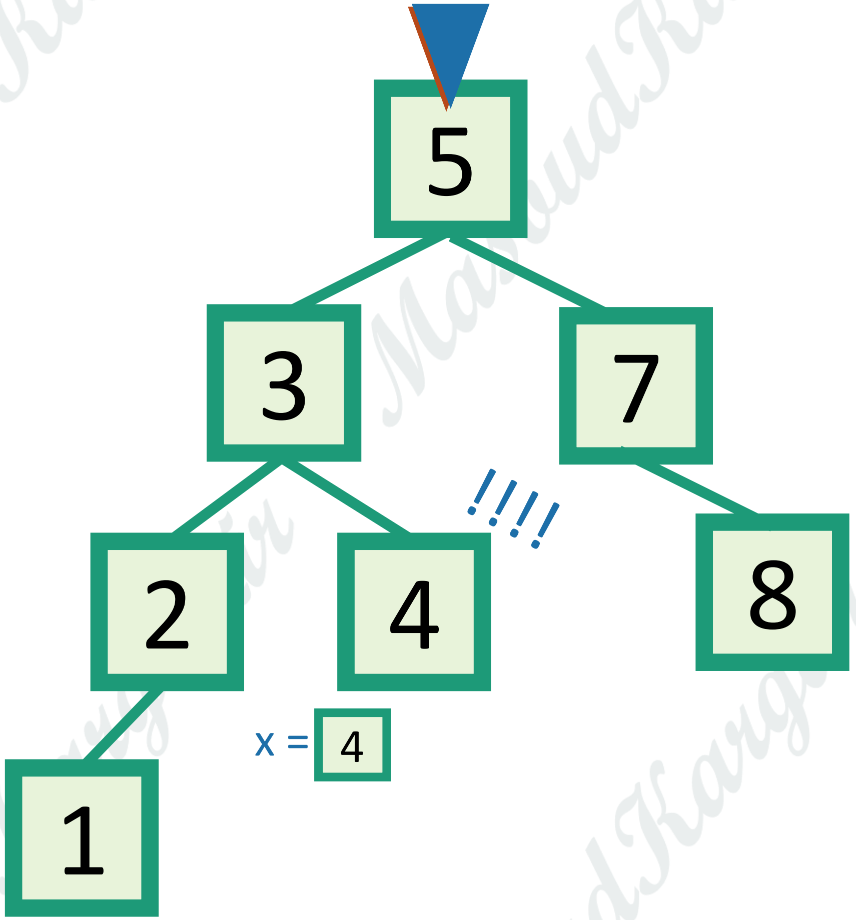
EXAMPLE: Search for 4.

EXAMPLE: Search for 4.5

- It turns out it will be convenient to **return 4** in this case
- (that is, **return** the last node before we went off the tree)

Write pseudocode (or actual code) to implement this!

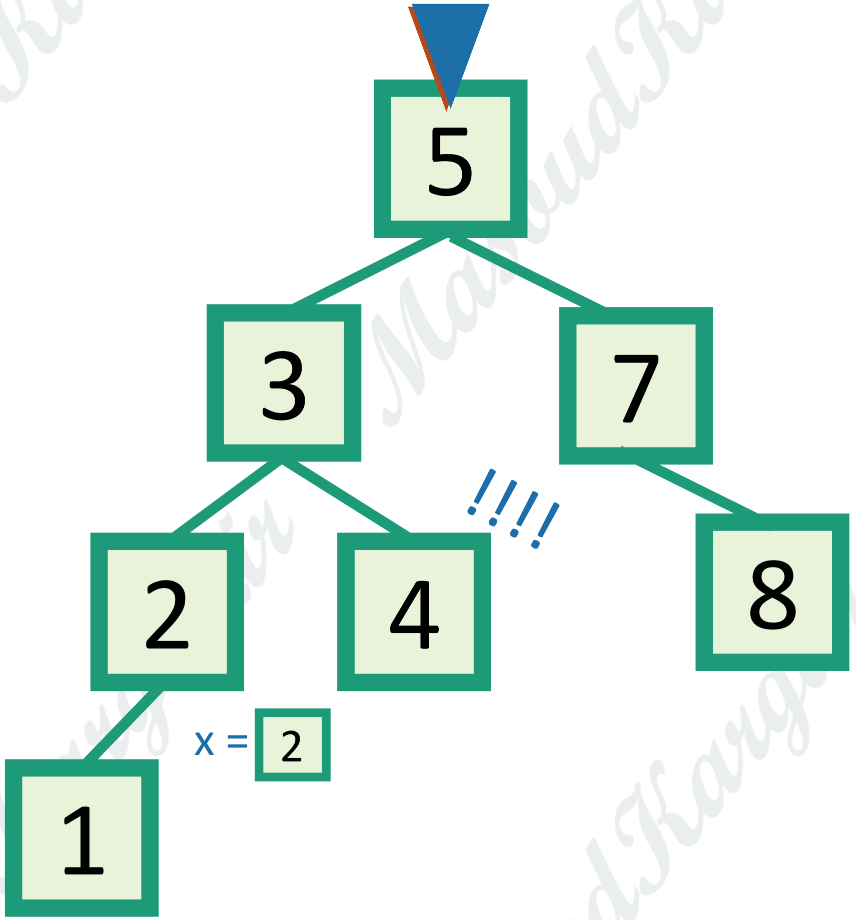
INSERT in a Binary Search Tree



EXAMPLE: Insert 4.5

- INSERT(key):
  - x = SEARCH(key)
  - if key > x.key:
    - Make a new node with the correct key, and put it as the right child of x.
  - if key < x.key:
    - Make a new node with the correct key, and put it as the left child of x.
  - if x.key == key:
    - **return**

DELETE in a Binary Search Tree

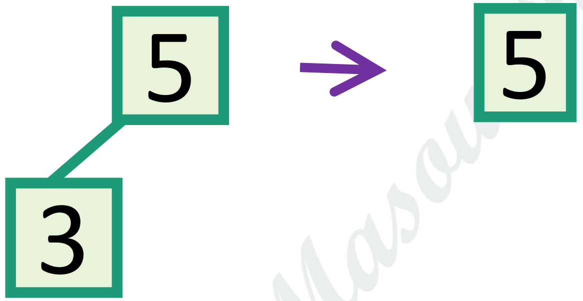


EXAMPLE: Delete 2

- DELETE(key):
  - x = SEARCH(key)
  - **if** x.key == key:
    - ....delete x....

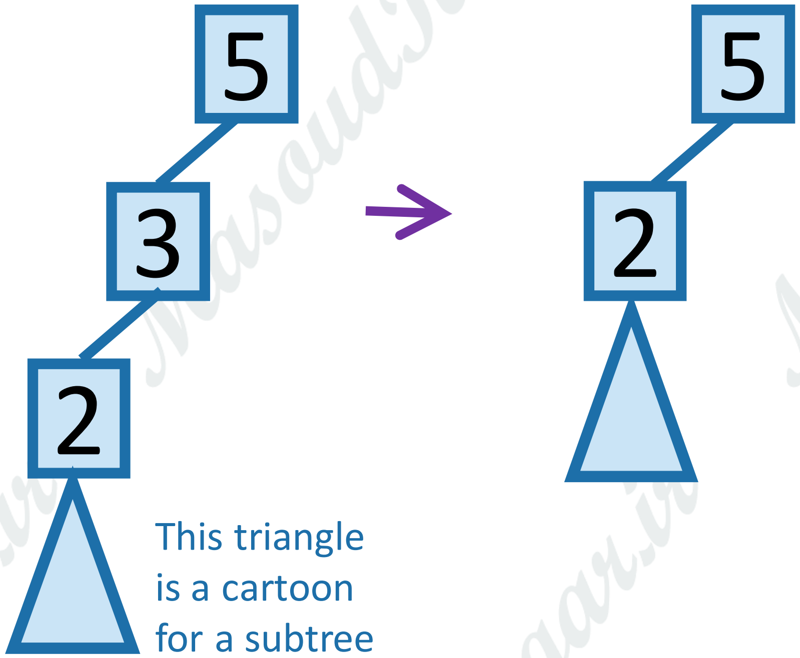
This is a bit more complicated...

DELETE in a Binary Search Tree
several cases (by example)
say we want to delete 3



Case 1: if 3 is a leaf,
just delete it.

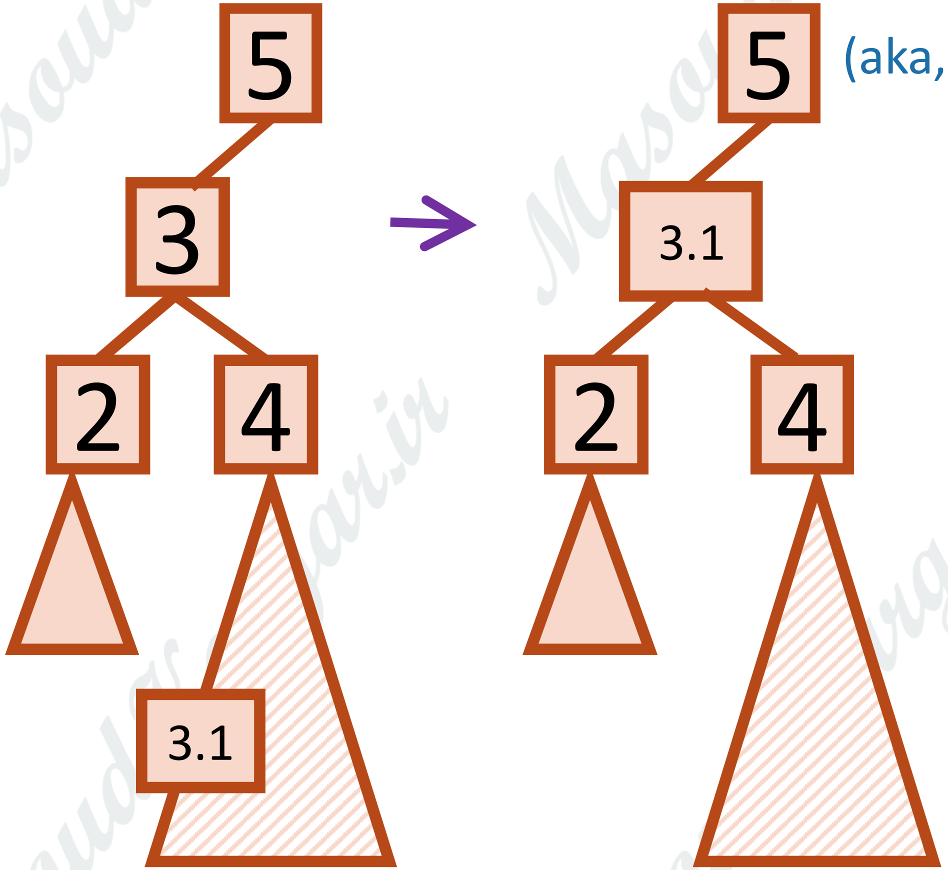
We won't write down pseudocode for this – try to do it yourself!



Case 2: if 3 has just one child,
move that up.

DELETE in a Binary Search Tree ctd.

Case 3: if 3 has two children,
replace 3 with it's immediate successor.

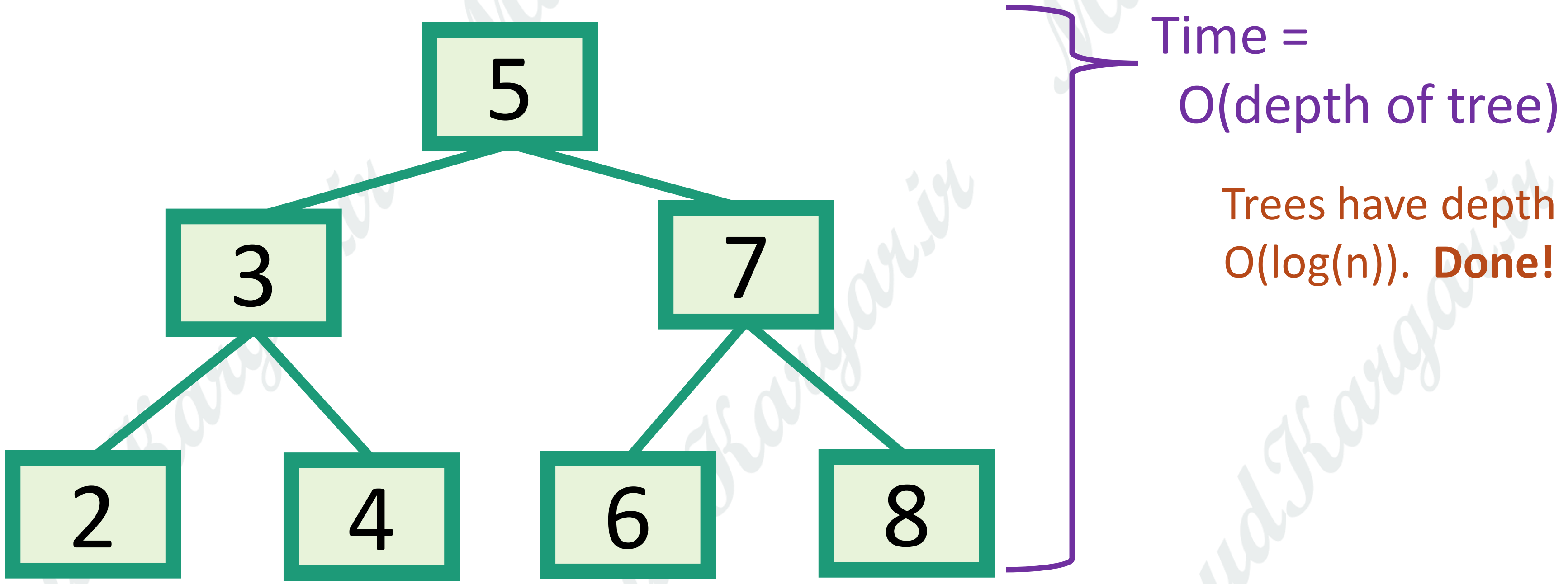


(aka, next biggest thing after 3)

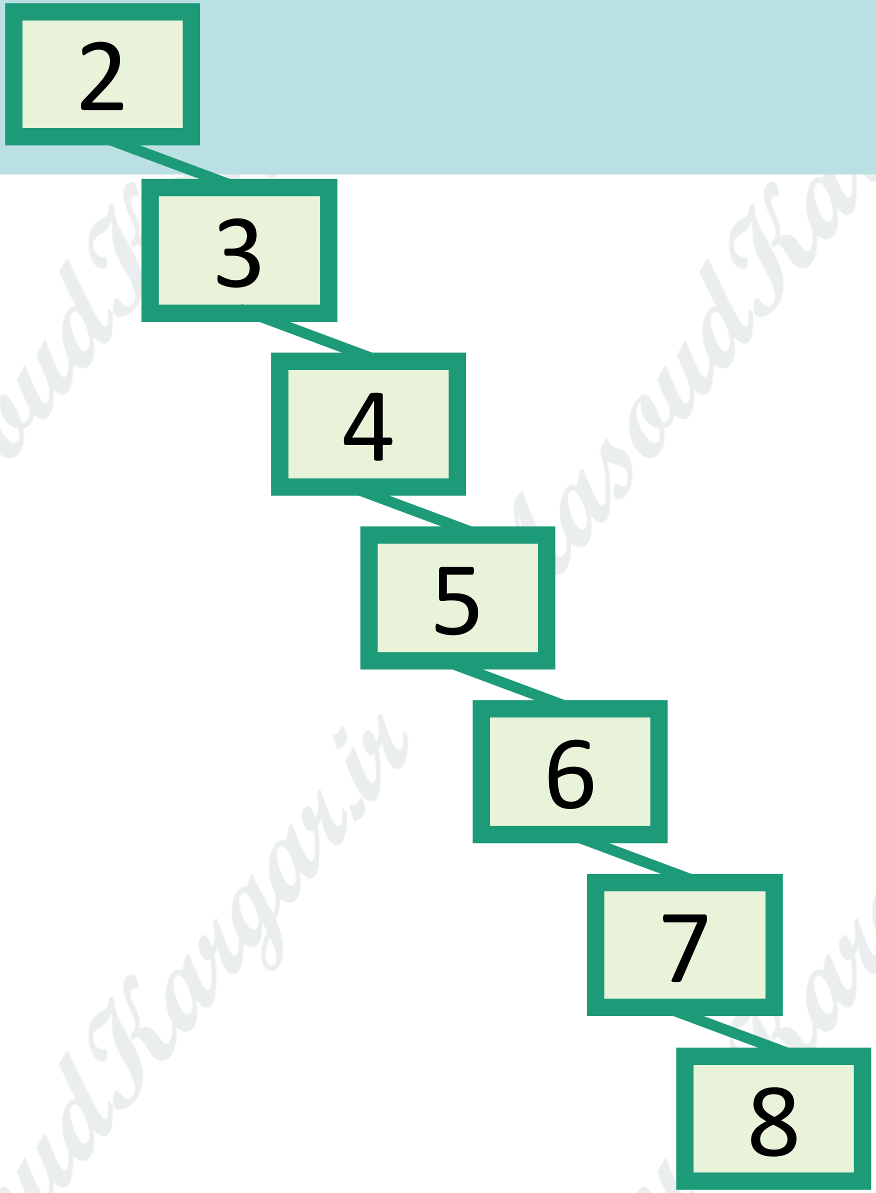
- How do we find the immediate successor?
  - SEARCH(3.right, 3)
- How do we remove it when we find it?
  - Run DELETE for one of the previous two cases.
- Wait, what if it's **THIS** case? (Case 3).
  - It's not.

How long do these operations take?

- SEARCH is the big one.
- Everything else just calls SEARCH and then does some small O(1)-time operation.



Wait...



- This is a valid binary search tree.
- The version with n nodes has depth n, not O(log(n)).

Could such a tree show up?

In what order would I have to insert the nodes?

Inserting in the order 2,3,4,5,6,7,8 would do it.

So this could happen.

What to do?

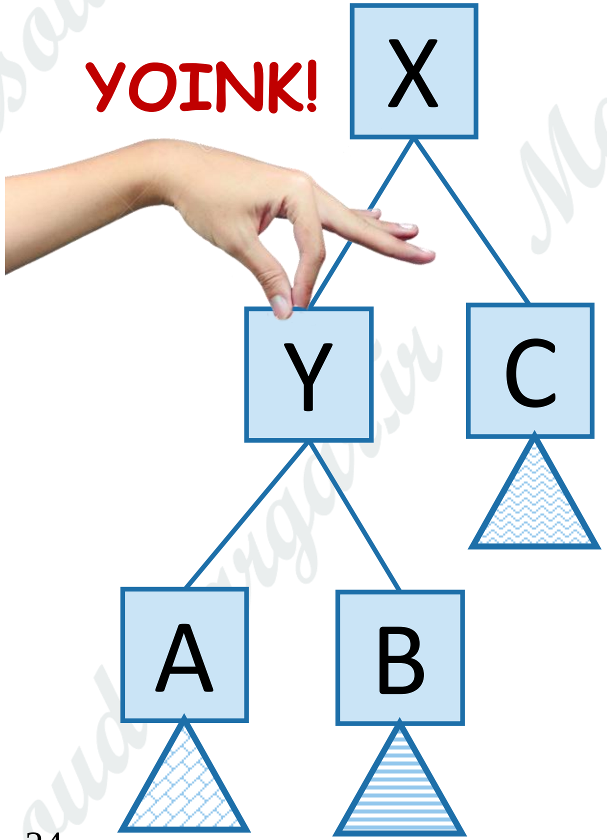
- Keep track of how deep the tree is getting.
- If it gets too tall, re-do everything from scratch.
  - At least Ω(n) every so often….

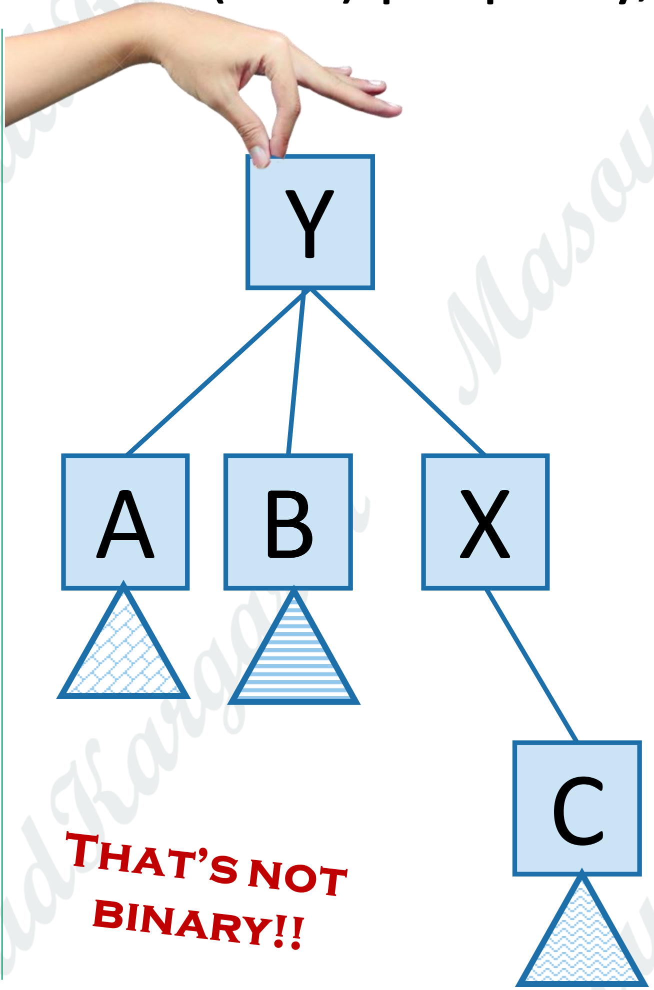
How often is "every so often" in the worst case? It's actually pretty often!

- Other ideas?

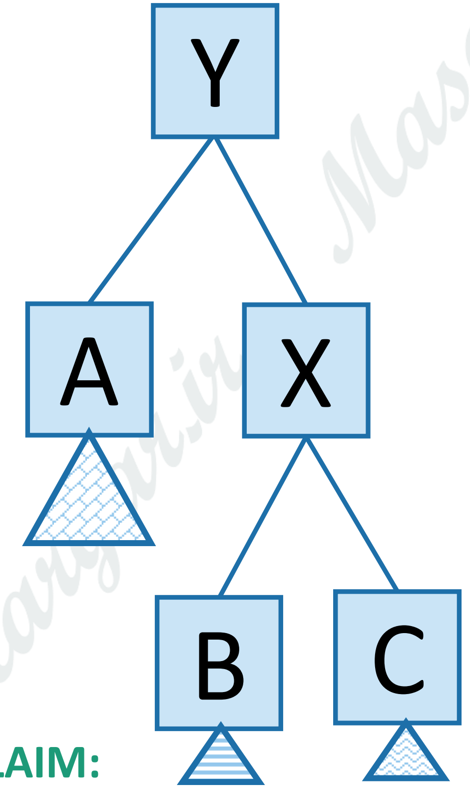
Idea 1: Rotations

- Maintain Binary Search Tree (BST) property, while moving stuff around.



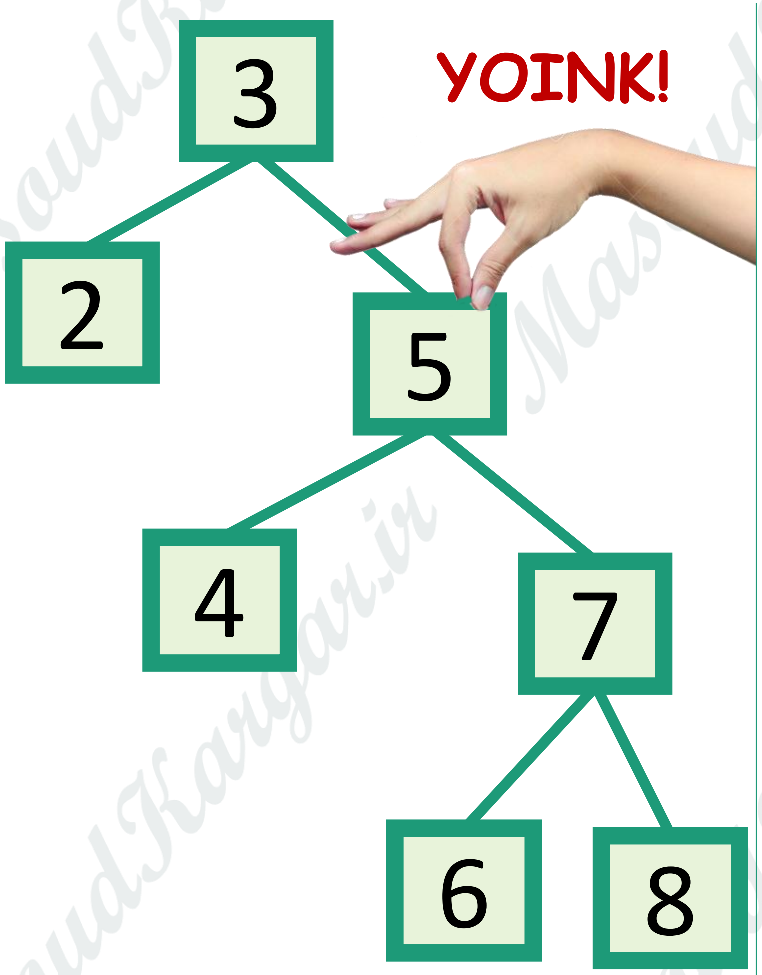


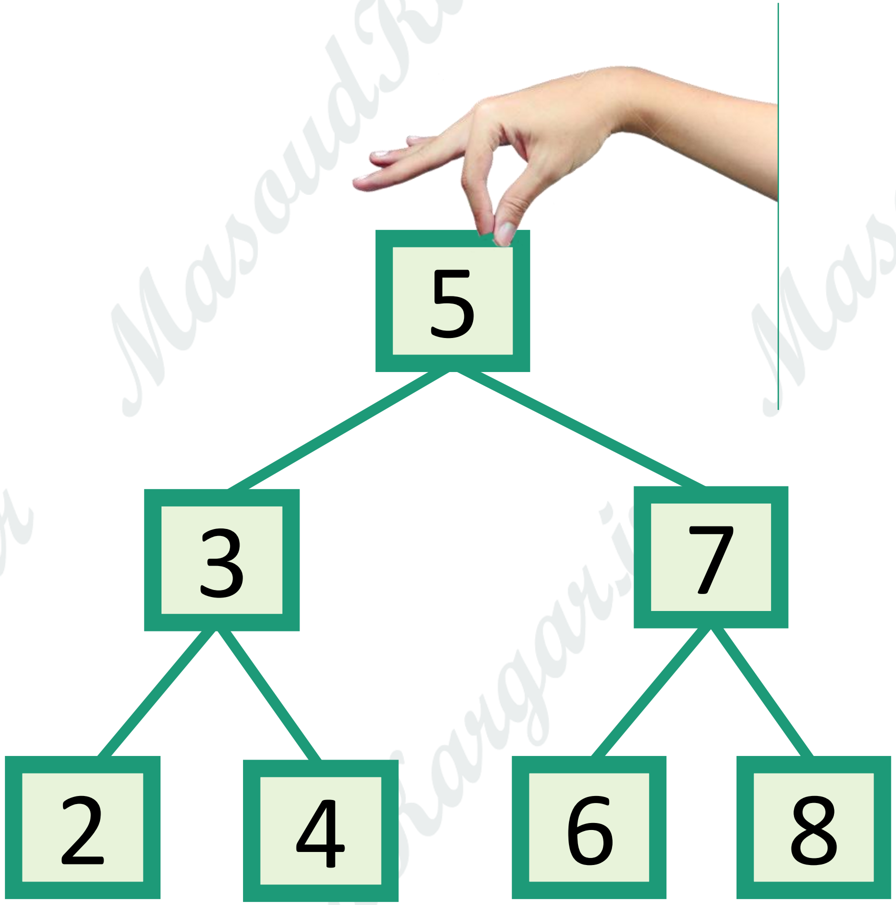
No matter what lives underneath A,B,C,
this takes time O(1). (Why?)



CLAIM:
this still has BST property.

This seems helpful





Does this work?

- Whenever something seems unbalanced, do rotations until it's okay again.

Even for me this is pretty vague. What do we mean by "seems unbalanced"? What's "okay"?

Idea 2: have some proxy for balance

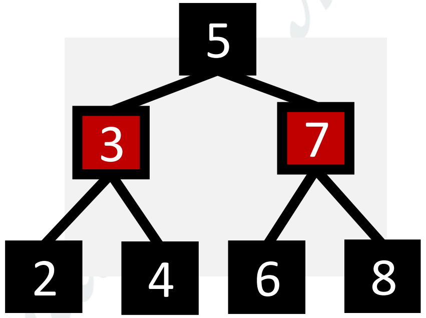
- Maintaining perfect balance is too hard.
- Instead, come up with some proxy for balance:
  - If the tree satisfies [SOME PROPERTY], then it's pretty balanced.
  - We can maintain [SOME PROPERTY] using rotations.

There are actually several ways to do this, but today we'll see…

Red-Black Trees

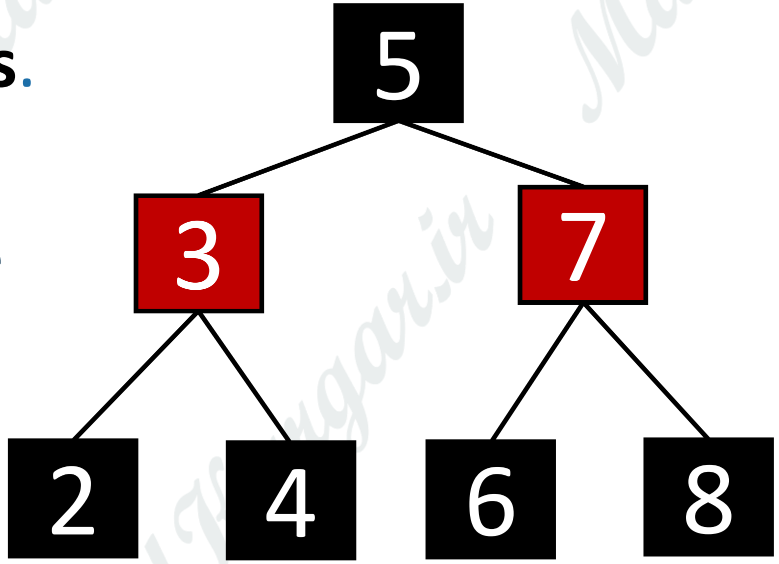
- A Binary Search Tree that balances itself!
- No more time-consuming by-hand balancing!
- Be the envy of your friends and neighbors with the time-saving…

Maintain balance by stipulating that **black nodes** are balanced, and that there *aren't too many* **red nodes**.

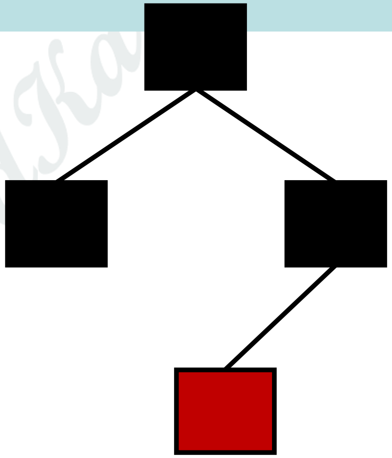


Red-Black Trees
these rules are the proxy for balance

- Every node is colored **red** or **black.**
- The root node is a **black node**.
- NIL children count as **black nodes**.
- Children of a **red node** are **black nodes**.
- For all nodes x:
  - all paths from x to NONE's have the same number of black nodes on them.



Examples(?)



- Every node is colored **red** or **black.**
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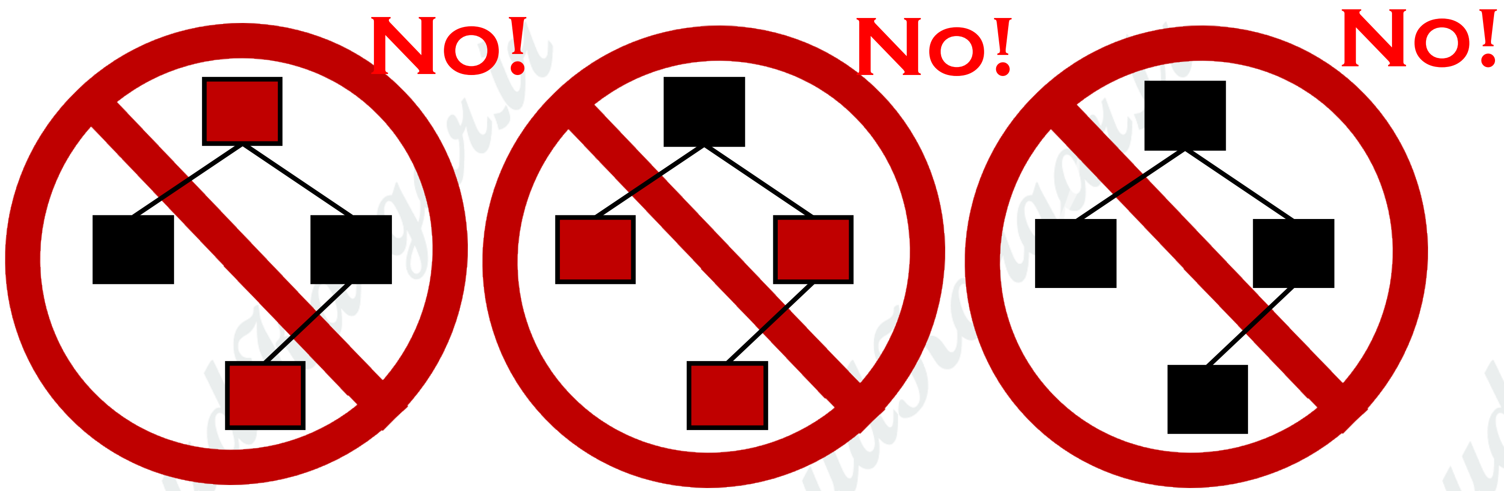
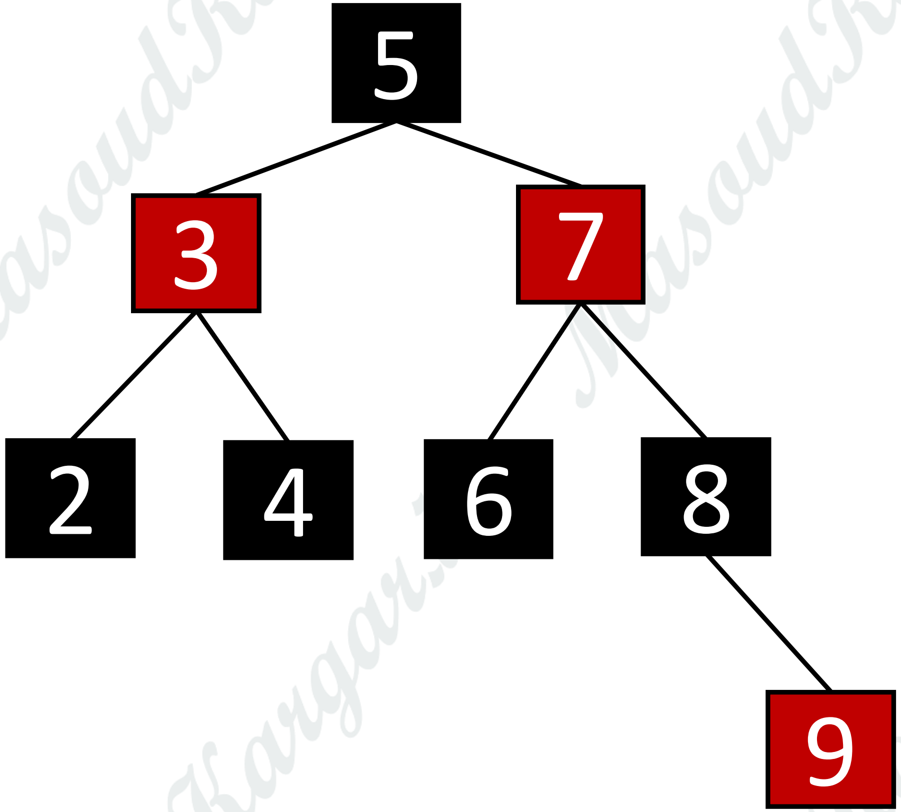


Why????????

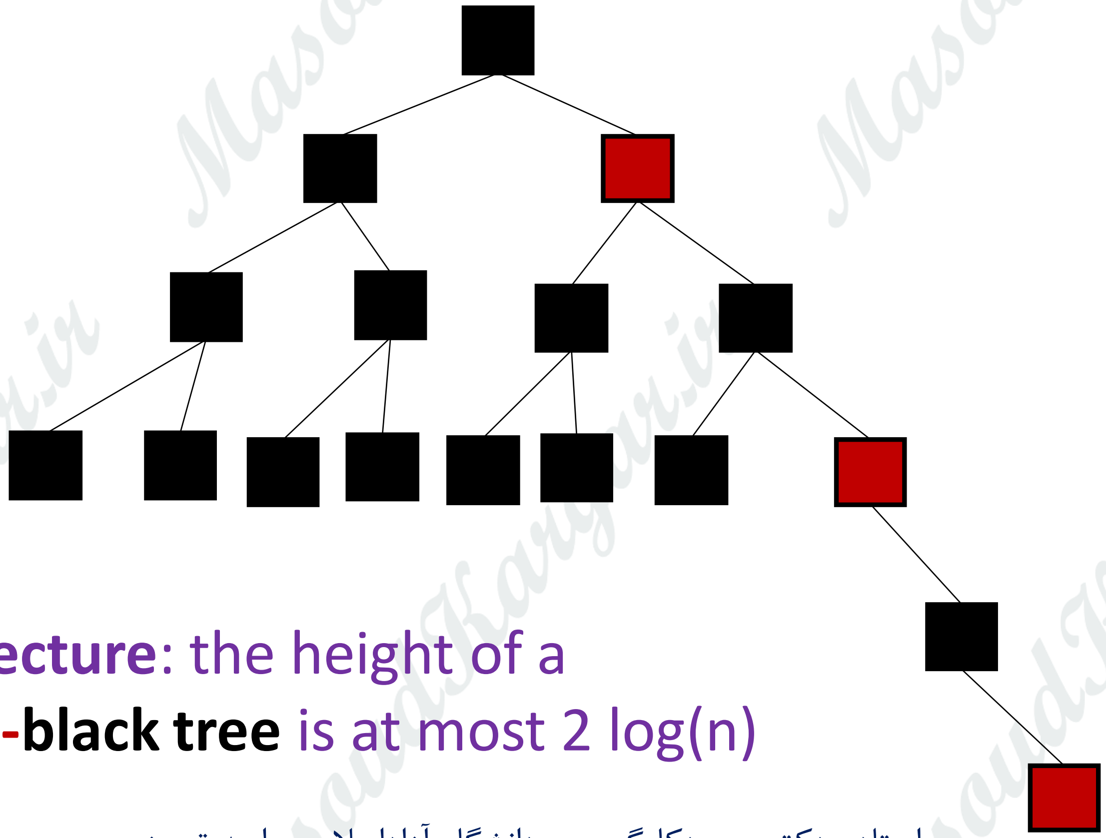
- This is pretty balanced.
  - The **black nodes** are balanced
  - The **red nodes** are "spread out" so they don't mess things up too much.

- We can maintain this property as we insert/delete nodes, by using rotations.



This is "pretty balanced"

- To see why, intuitively, let's try to build a Red-Black Tree that's unbalanced.



Let's build some intuition!

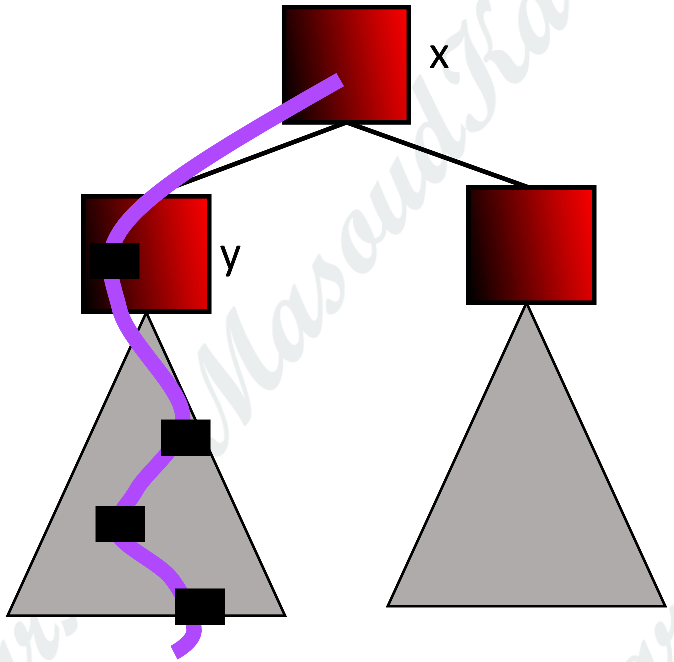
One path could be twice as long as all the others if we pad it with red nodes.

Conjecture: the height of a **red-black tree** is at most 2 log(n)

That turns out the be basically right.

[proof sketch]

- Say there are b(x) black nodes in any path from x to NONE.
  - (including x).
- **Claim:**
  - Then there are at least $2^{b(x)} - 1$ nodes in the subtree underneath x.
- [Proof by induction – on board if time]



Then:

$$n \geq 2^{b(root)} - 1$$

$$\geq 2^{height/2} - 1$$

using the **Claim**

b(root) >= height/2 because of RBTree rules.

Rearranging:

$$n + 1 \geq 2^{height/2} \Rightarrow height \leq 2\log(n+1)$$

Okay, so it's balanced...
...but can we maintain it?

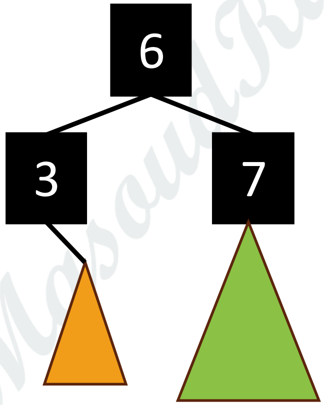
- Yes!

- For the rest of lecture:
  - sketch of how we'd do this.

- See CLRS for more details.

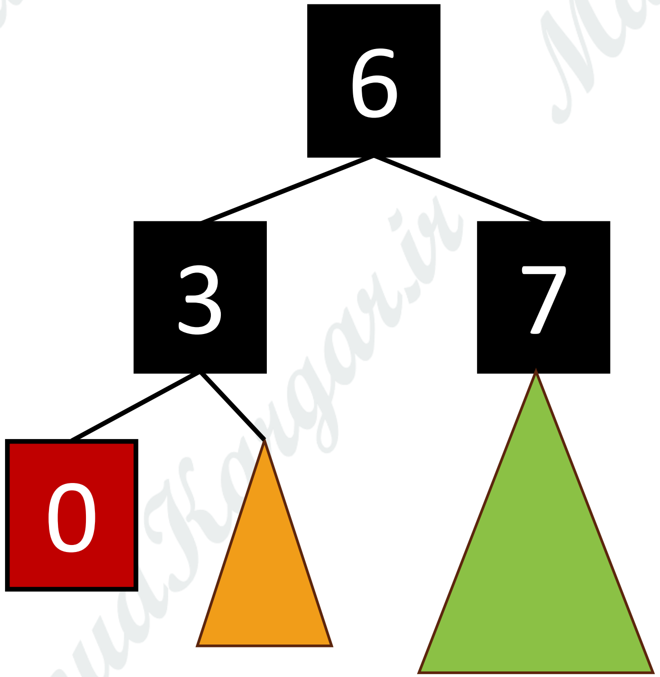
Inserting into a Red-Black Tree

- Make a new **red node.**
- Insert it as you would normally.



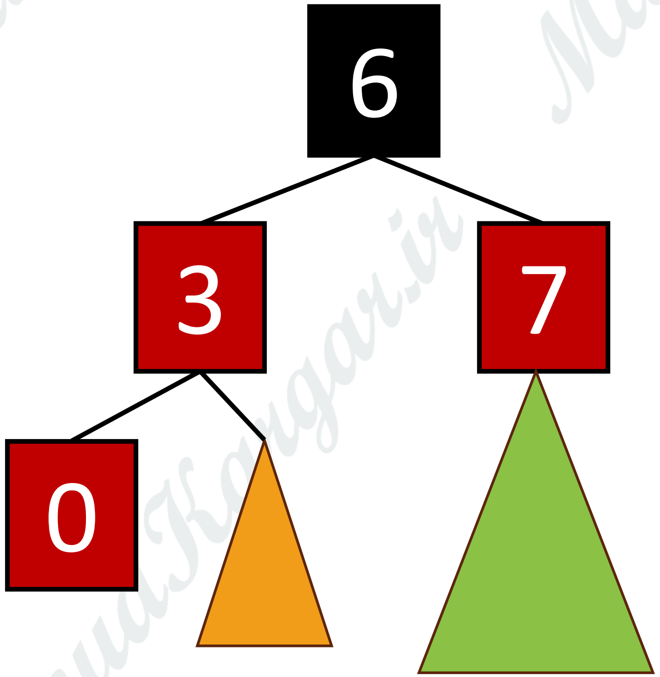
What if it looks like this?

Example: insert 0



Inserting into a Red-Black Tree

- Make a new **red node.**
- Insert it as you would normally.
- Fix things up if needed.



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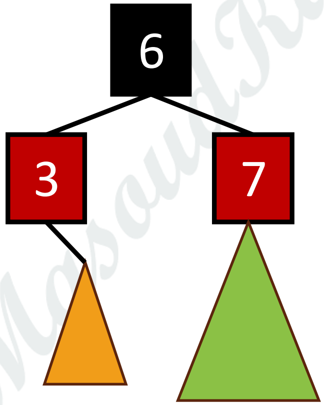

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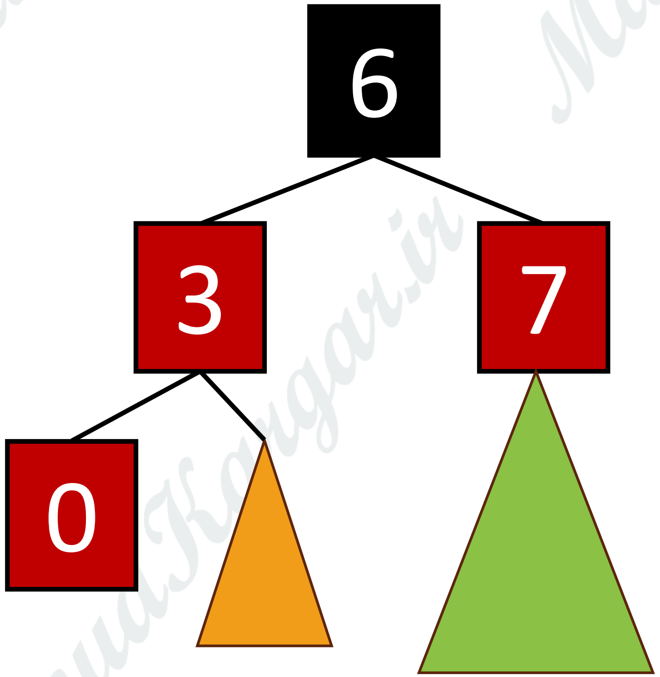
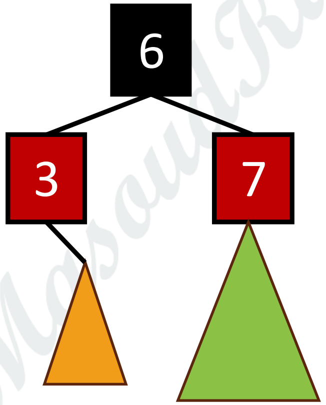


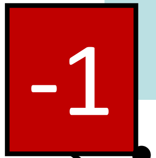
Example: insert 0

Can't we just insert 0 as a **black node?**



What if it looks like this?

Inserting into a Red-Black Tree



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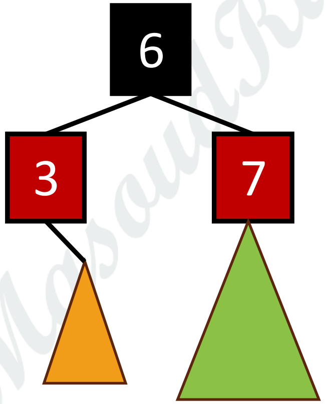
Example: insert 0



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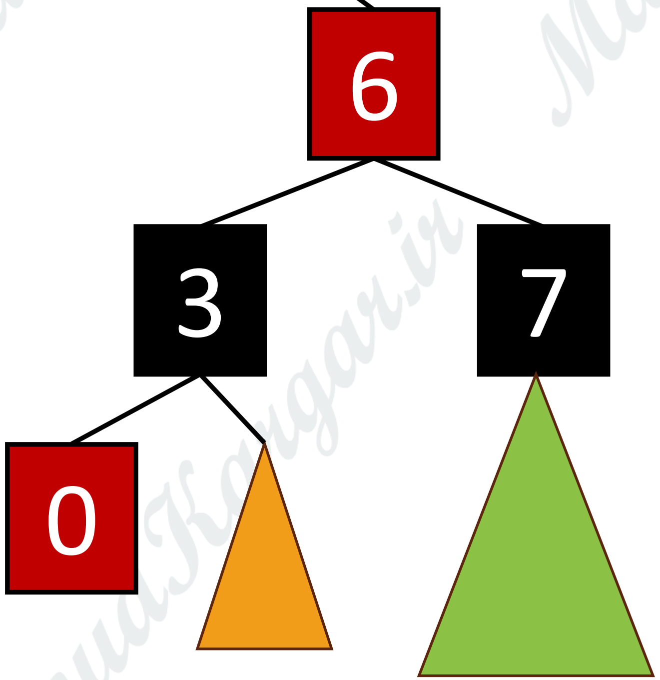
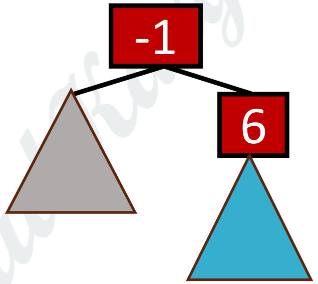


- Instead recolor like this.
- Need to argue:
  - RB-Tree properties still hold.

- What about the red root?
  - if 6 is actually the root, color it black.
  - Else, recursively re-color up the tree.

Now the problem looks like this, where I'm inserting



Inserting into a Red-Black Tree

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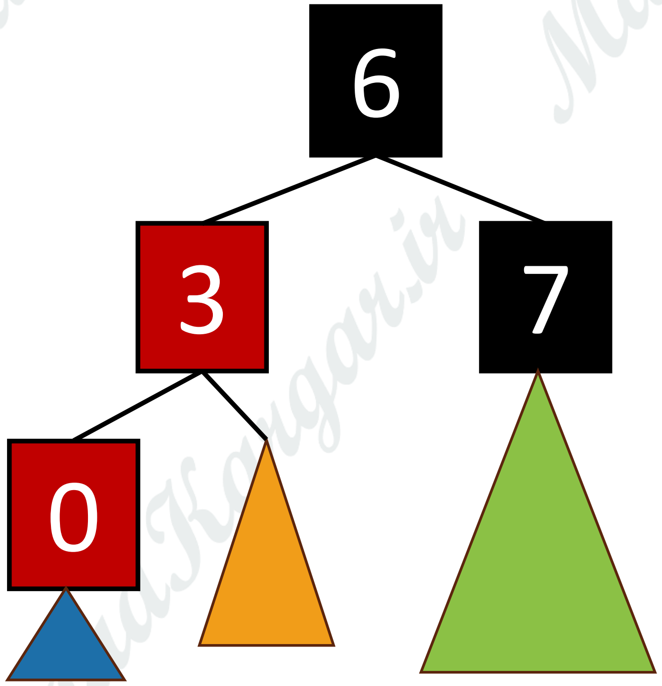
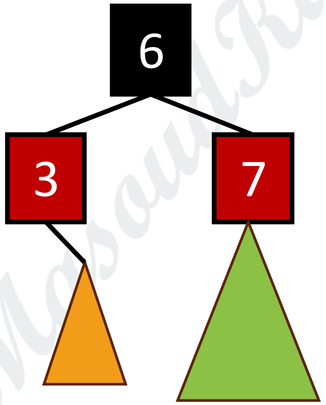


Example: Insert 0.

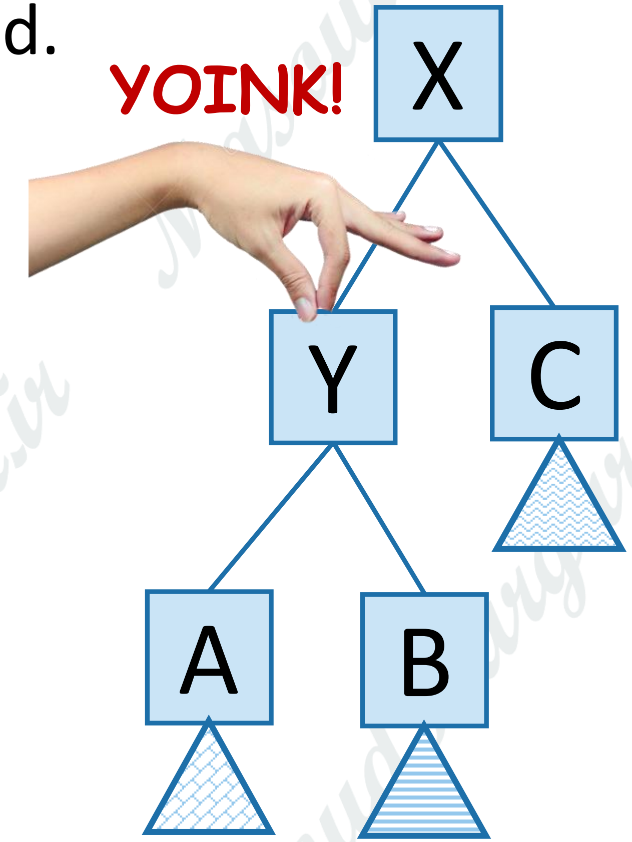


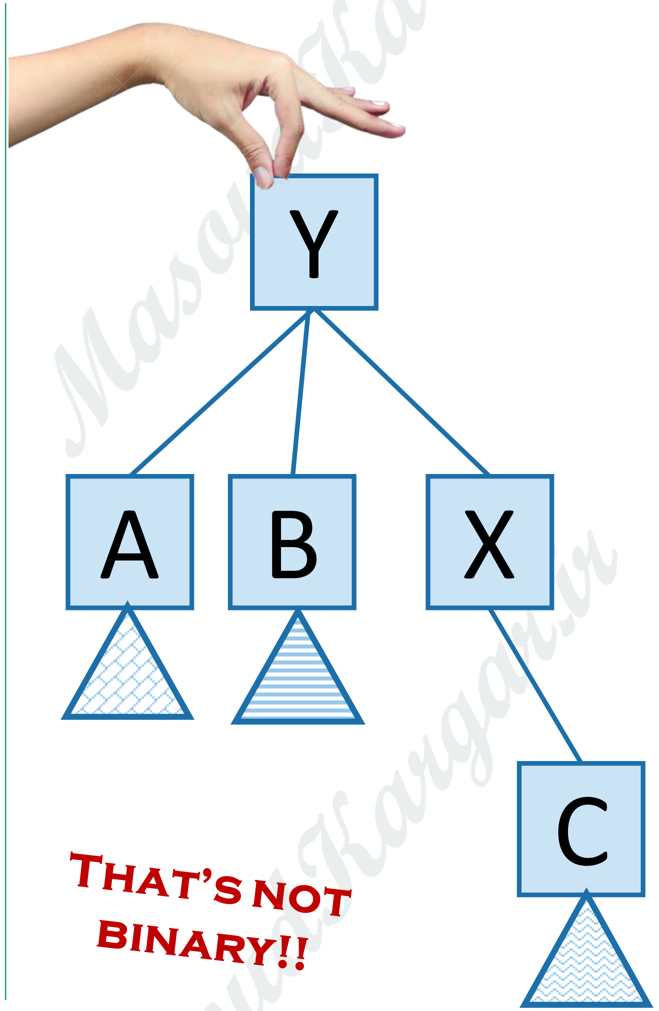
What if it looks like this?

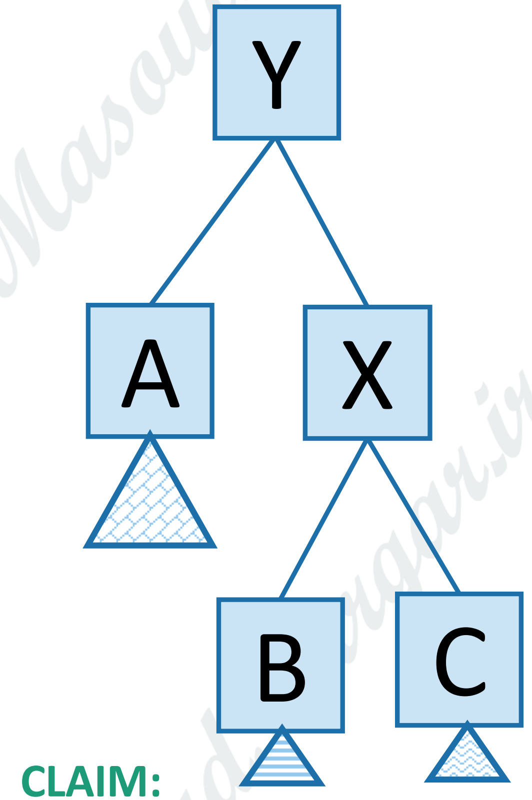
- Actually, **this can't happen?**
- It might happen that we just turned 0 red from the previous step.
- Or it could happen if 7 is actually NIL.

Recall Rotations

- Maintain Binary Search Tree (BST) property, while moving stuff around.







Inserting into a Red-Black Tree

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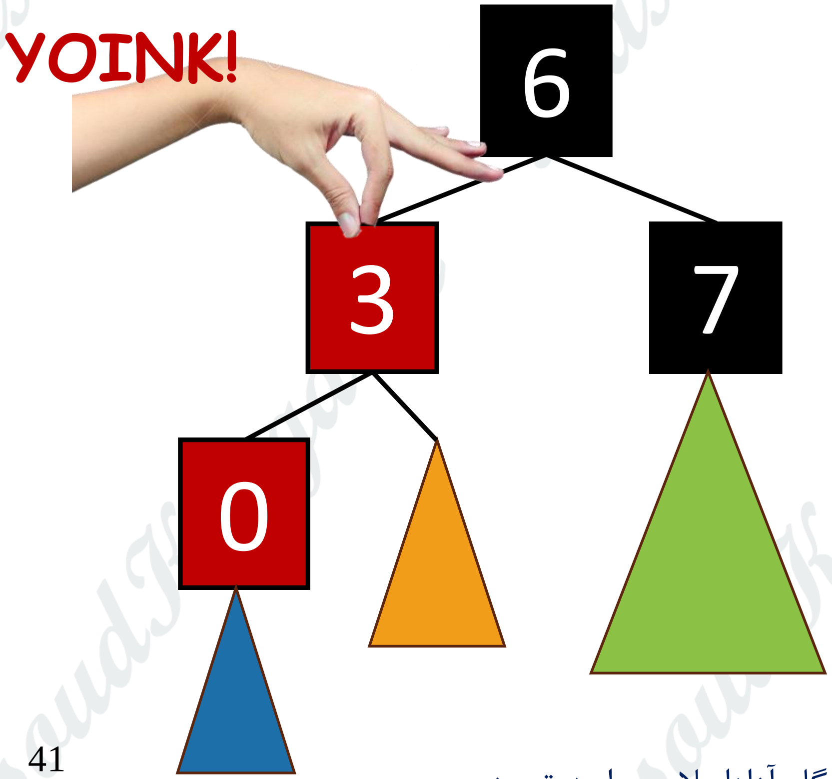
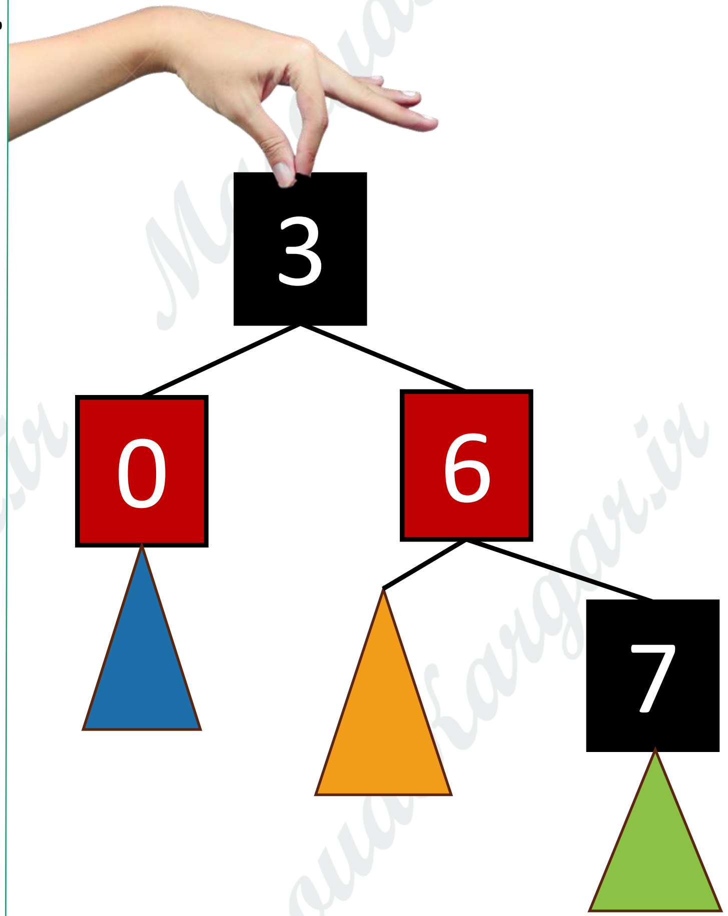
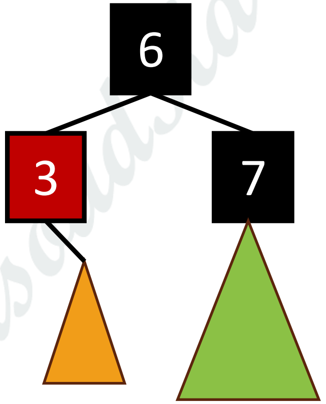
- Insert it as you would normally.
- Fix things up if needed.







What if it looks like this?

Need to argue that if RB-Tree property held before, it still does.

That's basically it

That's basically it

- A few things still left to check for INSERT!
  - Anything else that might happen looks basically like what we just did.
  - Formally dealing with the recursion.
  - You check these! (or see CLRS)
- DELETE is similar.

The punchline:

- Red-Black Trees always have height at most 2log(n+1).
- As with general Binary Search Trees, all operations are O(height)
- So all operations are O(log(n)).

Conclusion: The best of both worlds

	Sorted Arrays	Linked Lists	Balanced Binary Search Trees
Search	O(log(n))	O(n)	O(log(n))
Insert/Delete	O(n)	O(1)	O(log(n))

Recap

- Balanced binary trees are the best of both worlds!
- But we need to keep them balanced.
- **Red-Black Trees** do that for us.
  - We get O(log(n))-time INSERT/DELETE/SEARCH
  - Clever idea: have a proxy for balance

Next time

- Hashing!

قدردانی